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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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be a polynomial with integral co-efficients mod. p. Every polynomial mod. p can be reduced mod. $F(x)$ to a polynomial of degree $\leqslant n-1$. Thus:

$$f(x) = a_{11} + a_{21}x + \dots + a_{n1}x^{n-1} \text{ mod. } (p, F(x))$$

$$xf(x) = a_{12} + a_{22}x + \dots + a_{n2}x^{n-1} \text{ mod. } (p, F(x))$$

$$\vdots$$

$$x^{n-1}f(x) = a_{1n} + a_{2n}x + \dots + a_{nn}x^{n-1} \text{ mod. } (p, F(x))$$

Hence

$$f(x)\{x_1 + x_2x + \dots + x_nx^{n-1}\} = y_1 + y_2x + \dots + y_nx^{n-1} \text{ mod. } (p, F(x))$$

where $\quad y_i = a_{i1}x_1 + a_{i2}x_2 + \dots \dots a_{in}x_n$ (mod. p)

Thus multiplication of the residue-classes mod. $(p, F(x))$ by $f(x)$ is equivalent to a linear transformation mod. p, of the matrix $|a_{ij}|$. Now if det. $|a_{ij}| = 0$ mod. p, then we can find integers $k_1, k_2 \dots k_n$ not all divisible by p, so that

$$y_i = a_{i1}k_1 + a_{i2}k_2 + \dots + a_{in}k_n = 0 \text{ mod. } p, \quad (i = 1, 2, \dots n).$$

Hence $\quad f(x)\{k_1 + k_2x + \dots + k_nx^{n-1}\} = 0$ mod. $(p, F(x))$,

or is divisible by $F(x)$ mod. p; but $k_1 + k_2x \dots + k_nx^{n-1}$ is at most of the $(n-1)$th degree, and so can not be divisible by $F(x)$ mod. p. Hence $f(x)$ and $F(x)$ must have a common polynomial factor mod. p. We shall suppose therefore that $f(x)$ and $F(x)$ are mutually prime mod p, so that det $|a_{ij}| \neq 0$ mod. p. Let us use the symbol $\left(\dfrac{f(x)}{p, F(x)}\right)$ as in III to denote the class (± 1) of the permutation of the residue-classes mod. $(p, F(x))$ produced by multiplying them all by $f(x)$. From our lemma it follows that this class is also equal to the value of the Legendre-symbol

$$\left(\frac{\det |a_{ij}|}{p}\right);$$

i.e. $$\left(\frac{f(x)}{p, F(x)}\right) = \left(\frac{\det |a_{ij}|}{p}\right).$$

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A more convenient form may be given to this result. Let $\Delta(F, f)$ denote Sylvester's dialytic eliminant of the polynomials $f(x)$, $F(x)$, Writing:

$$f(x) = b_0 + b_1 x + \dots + b_m x^m \ (m > = \text{ or } < n)$$
$$F(x) = a_0 + a_1 x + \dots + a_n x^n,$$

$\Delta(f, F)$ is the determinant of order $m + n$,

$$\begin{vmatrix} b_0 & b_1 & \dots & \dots & b_m & 0 & \dots & 0 \\ 0 & b_0 & \dots & \dots & \dots & b_m & \dots & 0 \\ 0 & 0 & b_0 & \dots & \dots & \dots & \dots & b_m \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_0 & a_1 & \dots & \dots & a_n & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_0 & \dots & \dots & \dots & a_n \end{vmatrix}$$

Since $f(x)$, $F(x)$ have been assumed to be mutually prime mod. p $\Delta(f, F) \not\equiv 0$ mod. p.

In the determinant $\Delta(f, F)$, by adding suitable multiples of the last m rows to the first n rows, we can clearly make the elements common to the first n rows and the last m columns, equal to zero mod. p. When we have done this, it will be clear that the elements common to the first n rows and first n columns, will form the matrix $|a_{ij}|$.

Hence:

$$\Delta(f, F) = \text{Det } |a_{ij}| \,.\, a_n^m \ (\text{mod. } p).$$

$$\therefore \quad \left(\frac{\det |a_{ij}|}{p}\right) = \left(\frac{\Delta(f, F)}{p}\right) \cdot \left(\frac{a_n}{p}\right)^m$$

Hence $$\left(\frac{f(x)}{p, F(x)}\right) = \left(\frac{\Delta(f, F)}{p}\right) \cdot \left(\frac{a_n}{p}\right)^m$$

Similarly $$\left(\frac{F(x)}{p, f(x)}\right) = \left(\frac{\Delta(F, f)}{p}\right) \left(\frac{b_m}{p}\right)^n$$

Multiplying these equations together and noting that

$$\Delta(F, f) = (-1)^{mn} \Delta(f, F),$$

we have the reciprocity theorem:

$$\left(\frac{f(x)}{p, F(x)}\right)\left(\frac{F(x)}{p, f(x)}\right) = (-1)^{mn}\left(\frac{a_n}{p}\right)^m\left(\frac{b_m}{p}\right)^n.$$

which holds whenever $f(x)$, $F(x)$ are mutually prime mod. p.

We have seen in III, that, when $F(x)$ is a prime polynomial, $f(x)$ is a quadratic residue or non-residue of $F(x)$ according as

$$\left(\frac{f(x)}{p, F(x)}\right) = +1 \text{ or } -1.$$

In the case when both $f(x)$ and $F(x)$ are prime polynomials mod. p, we can deduce by our reciprocity theorem, the quadratic character of $f(x)$ mod. $(p, F(x))$, from that of $F(x)$ mod. $(p, f(x))$, or *vice-versa*

For two mutually prime polynomials $f(x)$, $F(x)$ with unit highest co-efficients, (*i.e.*, for two *primary*, relatively prime polynomials $f(x)$, $F(x)$ mod. p) the reciprocity theorem takes the form:

$$\left(\frac{f(x)}{p, F(x)}\right)\left(\frac{F(x)}{p, f(x)}\right) = (-1)^{mn}.$$

[For the case in which each of $f(x)$, $F(x)$ is a prime polynomial, this gives the interesting theorem;

> *Two primary, prime polynomials $f(x)$, $F(x)$ of degrees m, n mod. p, are either each a quadratic residue of the other, or each a non-residue, of the other, when one at least of the numbers m, n is even; if m, n are both odd, then one, and only one of the two polynomials is a quadratic residue of the other*].

This is the most convenient form of the reciprocity theorem. The general theorem may be easily derived from this. To see this, let $f_1(x)$, $F_1(x)$ be two mutually prime polynomials mod. p, of degrees m, n. Write

$$f_1(x) = b_m f(x),\ F_1(x) = a_n F(x),$$

where $f(x)$, $F(x)$ are primary. Then

$$\left(\frac{f_1(x)}{p, F_1(x)}\right) = \left(\frac{f_1(x)}{p, F(x)}\right) = \left(\frac{b_m}{p, F(x)}\right)\left(\frac{f(x)}{p, F(x)}\right).$$

Similarly $\left(\dfrac{F_1(x)}{p, f_1(x)}\right) = \left(\dfrac{a_n}{p, f(x)}\right)\left(\dfrac{F(x)}{p, f(x)}\right).$

By the present theorem $\left(\dfrac{f(x)}{p, F(x)}\right)\left(\dfrac{F(x)}{p, f(x)}\right) = (-1)^{mn}.$

Hence $\left(\dfrac{f_1(x)}{p, F_1(x)}\right)\left(\dfrac{F_1(x)}{p, f_1(x)}\right) = (-1)^{mn}\left(\dfrac{b_m}{p, F(x)}\right)\left(\dfrac{a_n}{p, f(x)}\right).$

To determine $\left(\dfrac{b_m}{p, F(x)}\right)$, we note that the matrix $|a_{ij}|$ for the polynomial b_m has all its diagonal elements equal to b_m, and all remaining elements equal to 0. Hence

$$\left(\frac{b_m}{p, F(x)}\right) = \left(\frac{b_m}{p}\right)^n \text{ and } \left(\frac{a_n}{p, f(x)}\right) = \left(\frac{a_n}{p}\right)^m.$$

On substituting we get back the general form of the reciprocity theorem.

THE AUTOMORPHIC ISOGONAL TRANSFORMATIONS OF A PLANE CUBIC.*

By S. Audinarayanan.

§ 1. Introduction.

The subject of quadratic transformations and their relations to cubic curves has been studied by many writers, †; but so far as I am aware, the problem of finding all the automorphic isogonal transformations of a cubic curve does not appear to have been discussed. The object of this paper is to investigate this problem.

The general quadratic transformations ‡ that transform a cubic into itself are of two kinds. If we represent points on the curve by elliptic parameters u (mod 2ω, $2\omega'$), with

$$u_1 + u_2 + u_3 = 0 \pmod{2\omega, 2\omega'}$$

as the condition of collinearity, the two types of transformations will be expressed by the equations

$$u' \equiv -u + c \pmod{2\omega, 2\omega'}$$

and

$$u' \equiv u + c' \pmod{2\omega, 2\omega'}.$$

The transformation defined by the first type is involutoric, the join of corresponding points passing through a fixed point on the curve. The second type will also be an involutoric transformation if the constant c' is any one of the three half periods ω, ω', $\omega + \omega'$. There are no double points on the curve for the second type of transformation.

The main results of this paper are the following:—

(1) A cubic curve Γ admits of an automorphic isogonal transformation of the first kind if the vertex K of the pencil of

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* I am greatly indebted to Dr. R. Vaidyanathaswamy for guidance and criticism in the preparation of this paper.

† Vide *Ency. der. Math Wiss.* III 2, 1 p. 499.

‡ The quadratic transformation with a conic of fixed points, *i.e.*, of the inversion type, is not considered in this paper.

polar lines of the absolute* with respect to all members of the syzygetic pencil of cubics defined by Γ, lies on Γ.

(2) But if the cubic Γ admits of more than one automorphic isogonal transformation of the first kind, it admits three. In this case the absolute has the same polar line L with respect to all cubics of the syzygetic pencil defined by Γ. The condition for this is that there exists a cubic F in the pencil whose polar conics are out-polar to the absolute.

For the Euclidean case this amounts to the existence of a cubic F in the syzygetic pencil defined by Γ whose polar conics are rectangular hyperbolas. The Hessian cubic H of this cubic F is a circular cubic passing through its singular focus. H has been called the "focal cubic."

(3) As for the transformations of the second kind, a cubic in general admits of three automorphic isogonal transformations T_1, T_2, T_3; each of these transformations has one system of cotangential pairs of Γ for corresponding points, each corresponding to one of the three cubics Γ_1, Γ_2, Γ_3 that have Γ for Hessian.

The three cubics are called pro-Hessian cubics † of Γ and are, by definition, members of the syzygetic pencil defined by Γ.

(4) If the cubic Γ is the Hessian cubic H of F whose polar conics are out-polar to the absolute, then Γ admits of ∞^2 automorphic isogonal transformations of the second kind.

In the last section of this paper, for a given isogonal transformation, all cubics that are automorphic for that and admit besides two other automorphic isogonal transformations are determined.

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* The polar line L of the absolute conic Ω is defined as the locus of points which together with the absolute is in-polar to the given cubic Γ. It is also the locus of poles of conics out-polar to the absolute. If $\Gamma \equiv a^3_x = 0$, $\Omega \equiv u^2_\alpha = 0$, then the polar line of the absolute is $L \equiv a^2_\alpha \, a_x = 0$.

† The word ' pro-Hessian ' is due to Prof, H, S, White.

§ 2. Properties of the three systems of co-tangential point-pairs on a cubic.

Given a cubic Γ_1, let the polar conic of P with respect to Γ_1 be a pair of lines, through P′; then the polar conic of P′ with respect to Γ_1 is a pair of lines through P. The locus of such points P, P′ is the Hessian cubic Γ of Γ_1. This cubic Γ is also the Jacobian of the net of polar conics of Γ_1. By the known property of the Jacobian, pairs of points related as P, P′ on Γ are conjugate points with respect to all the conics of the net, *i.e*, all the polar conics of Γ_1. We know that if two pairs of opposite vertices of a quadrilateral are conjugate pairs with respect to a conic, so is the third pair of vertices.† Hence,

THEOREM 1. If (P, P′), (Q, Q′) are conjugate pairs of points on the cubic Γ, then the points of intersection (R, R′) of (PQ, P′Q′) and (P′Q, PQ′) form a conjugate pair and hence lie on the cubic Γ.

To show that the tangents at such pairs of points meet on the curve, suppose Q and R to be consecutive points on the curve Γ; then their conjugate points Q′, R′ are also consecutive points on Γ, Hence by theorem 1 we have

THEOREM 2. The tangents at conjugate points Q, Q′ of a cubic Γ meet at a point P on the curve and QQ′ meets the curve again at P′, the conjugate of P.

Thus conjugate point pairs on a cubic Γ are corresponding points (co-tangential pairs) of Γ. Therefore if the four tangents from K, a point on Γ, be drawn to it, the points of contact P_1, P_2, P_3, P_4 are two conjugate pairs (P_1, P_2) (P_3, P_4) and the lines P_1P_2, P_3P_4 meet at A on the cubic Γ; also A and K are conjugate points. But the four points of contact can be grouped in three ways so that there are three systems of conjugate (co-tangential) points on every cubic. So a cubic can be considered as the Jacobian of three different nets of conics or in other words as the Hessian of three cubics Γ_1, Γ_2, Γ_3 defined by the three nets of conics. The three cubics are termed the pro-Hessians of Γ and

† Hesse's theorem. This may be proved easily by projecting the conic into a circle and one vertex into the centre.

are in the syzygetic pencil defined by Γ. The three systems of corresponding pairs of Γ define the three involutions I_1, I_2, I_3 of the second type of transformation corresponding to the three half-periods ω, ω' and $\omega + \omega'$. The corresponding points of K in the three systems are the vertices of the harmonic triangle of $P_1P_2P_3P_4$ and hence are points on Γ. Thus

THEOREM 3. The vertices of the harmonic triangle ABC of a contact quadrangle $P_1P_2P_3P_4$ of tangents from a point K on Γ to it, lie on Γ and K, A, B, C form another co-tangential quadrangle of Γ.

We shall now establish an important property of the co-tangential quadrangle $P_1P_2P_3P_4$ inscribed in Γ whose common tangential point is K. If P_1P_2 be a pair of conjugate points on the cubic Γ, and Q any point on it, then since the polar of P_1 with respect to the degenerate polar conic through Q is QP_2, the lines QP_1, and QP_2 are harmonic conjugates with respect to the degenerate conic through Q. Hence the pairs of lines joining Q to pairs of corresponding points are in involution. Now the polar conic of K with respect to Γ_1 is a line pair through A and this line pair is harmonically conjugate to AP_1, AP_2 and also AP_3, AP_4. Hence the line pair is (P_1P_2, P_3P_4). But we know that the polar conic of K with respect to Γ passes through $P_1P_2P_3P_4$. Hence, we have the property of the syzygetic pencil of Γ (*i.e.* the pencil defined by Γ and Γ_1).

THEOREM 4. The polar conics of a point K with respect to the members of a syzygetic pencil of cubics form a pencil through the four points of contact of tangents from K to that cubic of the pencil which passes through K.

The three line-pairs through the points $P_1P_2P_3P_4$ are the three polar conics of K with respect to the three pro-Hessian cubics $\Gamma_1\Gamma_2\Gamma_3$ of Γ in the pencil.

§ 3. The Quadratic transformations of a cubic into itself.

Let $P_1P_2P_3P_4$ be four points in a plane whose harmonic triangle is ABC.. If the polars of a point L with respect to all conics through $P_1P_2P_3P_4$ intersect in L′, then LL′ are related by an involutoric Cremona quadratic transformation which is completely determined by $P_1P_2P_3P_4$. The points ABC are singular points of this transformation

and correspond to indeterminate points on the opposite sides; also, the points $P_1P_2P_3P_4$ are self-corresponding points of the transformation.

This transformation admits of invariant cubic curves; for instance, we can determine one such invariant curve corresponding to each point K of the plane; for on each line of the plane there is a pair of corresponding points of the transformation and the locus of such corresponding points on lines through K is a cubic curve, "*the Jumelaire cubic*" *of K with respect to $P_1P_2P_3P_4$.** This cubic passes through ABC and through $P_1P_2P_3P_4$, the tangents at the last four points being KP_r $(r = 1, 2, 3, 4)$; also the Jumelaire cubic passes through K itself and through its corresponding point K′ and touches KK′ at K. Each of the ∞^2 Jumelaire cubics of $P_1P_2P_3P_4$ is by definition invariant for the quadratic transformation defined by $P_1P_2P_3P_4$.

Conversely, let K be any point on a cubic curve Γ, $P_1P_2P_3P_4$ the points of contact of tangents from K to Γ. By theorem (3), the harmonic triangle ABC of $P_1P_2P_3P_4$ is inscribed in Γ. The Jumelaire cubic of K with respect to $P_1P_2P_3P_4$ must pass through ABC and touch KP_r at P_r $(r = 1, 2, 3, 4)$ and so has more than nine points in common with Γ.

Thus the cubic Γ is the Jumelaire cubic of any of its points K with respect to the contact quadrangle of tangents from K.

Thus every cubic curve admits of ∞^1 automorphic transformations defined by inscribed quadrangles of the type $P_1P_2P_3P_4$ Pairs of points on the curve that correspond in this transformation are always collinear with a fixed point of the curve. These ∞^1 transformations may be called *the automorphic transformations of the first kind of* Γ.

But these do not exhaust all the involutoric quadratic transformations that the curve admits. For, we have seen that there are three involutions I_1, I_2, I_3 of co-tangential point pairs on the curve. Also any involution of point pairs on the curve must be either one of I_1, I_2, I_3 or must be such that the join of corresponding points passes through a fixed point K on the curve. Hence the involutoric automorphic quadratic transformations other than those of the first kind, must be those for

* Malet: *Transformations Birationalles*, page 96,

which the pairs of I_1, I_2, or I_3 are corresponding pairs. Since I_1, I_2, I_3 do not possess double points, an automorphic transformation of the first kind differs from one of the second kind in that the curve always contains the four fixed points of the former, but never any fixed point of the latter.

A quadratic transformation will transform Γ into a curve of degree higher than three, unless Γ contains the three singular points of it. We may therefore assume the singular points ABC of an automorphic quadratic transformation T, of the second kind to be points on Γ. Let u, v, w be the elliptic parameters of ABC (mod. 2ω, $2\omega'$) with $u_1+u_2+u_3 \equiv 0$ (mod 2ω, $2\omega'$) as the condition of collinearity; assume that the involution of corresponding points of T on the curve is the involution I_1. Let BC, CA, AB cut the curve again in A', B', C' and let A_1, B_1, C_1 be points neighbouring to ABC on the curve. A_1, A' are corresponding points in T so that AA'. BB', CC' are corresponding points of I_1. Hence in the elliptic parameter representation,

we have $$-v-w = u+\omega \pmod{2\omega, 2\omega'}$$

or $$u+v+w=\omega \pmod{2\omega, 2\omega'}.$$

This gives a necessary relation between the positions of the singular points of the transformation of the second kind.

Conversely, any three points ABC on the curve whose elliptic parameters satisfy the relation $2(u+v+w) = 0$ (mod 2ω, $2\omega'$) are singular points of an automorphic transformation T of the second kind. For the relation between the elliptic parameters of ABC implies that $A'B'C'$ lie on a line L and that there exists a conic S which touches the cubic at ABC. Let T be the transformation which has singular points at ABC and transforms L into S; let T transform Γ into the curve Γ'. Since Γ passes through ABC $A_1A'B_1$ $B'C_1C'$, Γ' must pass respectively through ABC $A'A_1$ $B'B_1$ $C'C_1$. Thus Γ' is a member of the pencil of cubics touching Γ at ABC and passing through $A'B'C'$. The two degenerate cubics of this pencil, one consisting of the sides of the triangle ABC and the other of L and S are evidently each transformed into itself by T. Also there is a cubic of this pencil which passes through a fixed point of this transformation and this must also be transformed into itself by T. Thus there are more than two cubics in the pencil that are automorphic for T. Hence T transforms every member of the pencil and

therefore Γ into itself. Thus in conjunction with the result for transformations of the first kind we have the following theorem.

There are ∞^1 automorphic transformations of the first kind each corresponding to a point K on the curve, having for fixed points the contact points of tangents from K to the curve, and three systems of ∞^2 automorphic transformations of the second kind each system correspondng to one of the co-tangential involutions on Γ.

§ 4. The isogonal transformations of the second kind.

The quadratic transformation defined by $P_1P_2P_3P_4$ is an *isogonal transformation* when the pair of cricular points is a corresponding pair of the transformation, or more generally, if we replace the circular points by a non-singular absolute conic Ω, if all conics through $P_1P_2P_3P_4$ are out-polar to the absolute; that is, if $P_1P_2P_3P_4$ is a self-polar quadrangle of Ω. To obtain all automorphic transformations of the second kind which are isogonal, let us consider the three pro-Hessian cubics Γ_1, Γ_2, Γ_3 of Γ. It has been seen that the point-pairs of the three co-tangential involutions I_1, I_2, I_3 on Γ are respectively conjugate points in regard to Γ_1, Γ_2, Γ_3. Hence it follows that the quadrangles which define the ∞^2 automorphic transformations of the second kind are constituted by the base points of a pencil of polar conics of Γ_1, Γ_2 or Γ_3. or what is the same thing, by the four poles of a line with respect to Γ_1, Γ_2 or Γ_3. To pick out the isogonal transformations from these, let L_r be the polar line of the absolute with respect to Γ_r $(r = 1, 2, 3)$ Then it is clear that L_r is the only line whose four poles with respect to Γ_r constitute a self-polar quadrangle Ω. Hence if T_r is the transformation defined by these four poles we have the result.

A cubic in general admits exactly three automorphic isogonal trans formations T_1, T_2, T_3 of the second kind; each of these transformations has the co-tangential pairs of one system for corresponding points.

This theorem will cease to be true in only one case, namely, when the absolute is in-polar to all the polar conics of one of the pro-hessian

cubics Γ_1, Γ_2, Γ_3 of Γ, say Γ_r*. In this case the transformations T_s, T_t ($r, s, t = 1, 2, 3$) are unique, but all transformations of the same system as T_r are isogonal transformations. If the absolute is a pair of points I, J, then Γ will admit ∞^2 isogonal transformations, only if I, J is a co-tangential pair on Γ; for a point-pair I, J in order to be in-polar to (*i.e.*, conjugate with respect to) all polar conics of Γ_r must be a co-tangential pair on Γ belonging to I_r. Hence the only cubics in the Euclidean plane that admit of ∞^2 isogonal transformations are the focal cubics † (*i.e.*, the circular cubics passing through their singular foci).

§ 5. The isogonal transformations of the first kind.

If the quadratic transformation defined by $P_1P_2P_3P_4$ is a transformation of Γ into itself of the first kind, then it was seen that Γ must be a Jumelaire cubic of some point K with respect to the quadrangle, so that P_1, P_2, P_3, P_4 are some four co-tangential points on Γ. If this transformation is to be isogonal, then $P_1P_2P_3P_4$ must be a self-polar quadrangle of the absolute conic Ω. Now this evidently amounts to two conditions on the quadrangle $P_1P_2P_3P_4$. Since there exists only ∞^1 inscribed co-tangential quadrangles of Γ, it follows that, *in general, a cubic does not admit isogonal transformations of the first kind into itself.*

We have seen that conics through $P_1P_2P_3P_4$ are polar conics of the common tangential point K with respect to the cubics of the syzygetic pencil determined by the given cubic Γ. If $P_1P_2P_3P_4$ determines an isogonal transformation of the cubic into itself, then the pencil of conics through $P_1P_2P_3P_4$ must be out-polar to the absolute conic Ω. Hence K must lie on the polar lines of the absolute with respect to every cubic of the syzygetic pencil. Conversely, if K is the vertex of the pencil of polar lines of the absolute with respect to the syzygetic pencil of cubics and KP_r ($r = 1, 2, 3, 4$) are tangents from K to that cubic Γ of the pencil which passes through K, then the contact quadrangle $P_1P_2P_3P_4$ of Γ determines an isogonal transformation of Γ into itself of the first kind. Hence,

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* This cannot happen for more than one of the cubics Γ_1, Γ_2, Γ_3 unless Γ is nodal.

† It will be seen in the next article that a cubic never admits of infinity of transformations of the first kind.

There exists one cubic in every syzygetic pencil of cubics which admits of an automorphic isogonal transformation of the first kind.

But an exceptional case occurs if the absolute has the same polar L in regard to every member of the syzygetic pencil of cubics. In this case there must exist one cubic F in the pencil with respect to which the polar line of the absolute is indeterminate. Then the absolute must be in-polar to all polar conics of F. For every point of intersection of L with the cubic Γ there must exist one isogonal transformation of the first kind transforming the cubic into itself; Hence there are three automorphic isogonal transformations for Γ and the same is true for every cubic of the syzygetic pencil. The fixed points of the three isogonal transformations thus determined, being contact points of tangents from three collinear points on Γ, form three quadrangles every two of which are in quadruple perspective from the vertices of the third, constituting the well-known Hesse's $(12_4, 16_3)$ configuration.*

Thus in a syzygetic pencil of cubics every cubic will admit of three automorphic isogonal transformations of the first kind when there exists a cubic F in the pencil whose polar conics are all out-polar to the absolute. In all other cases there is only one cubic in the pencil which admits of an automorphic isogonal transformation of the first kind.

§ 6. Jumelaire cubics admitting three isogonal automorphic transformations of the first kind.

We know that given an isogonal transformation defined by $P_1P_2P_3P_4$ there exists a net of Jumelaire cubics invariant for the transformation. We shall now determine all the cubics of this net that admit of two more automorphic transformations of the first kind. We have an interesting property of the net of Jumelaire cubics defined by $P_1P_2P_3P_4$, namely that,

* A discussion of and reference to this configuration will be found in the *Trans. Amer. Math. Soc.*, Vol 28, p. 502 and also in *Proc. Camb. Phil. Soc.* Vol. 21, p. 297.

The polar lines of the absolute conic Ω with respect to every cubic of the net of Jumelaire cubics defined by $P_1P_2P_3P_4$ all pass through a fixed point S which is the isogonal conjugate of the ortho-centre O* of the diagonal triangle ABC of $P_1P_2P_3P_4$.

For, suppose the equation of the absolute is

$$\alpha l^2 + \beta m^2 + \gamma n^2 + 2\alpha' mn + 2\beta' ln + 2\gamma' lm = 0$$

referred to the triangle ABC. The co-ordinates of $P_1P_2P_3P_4$ are then

$$(\pm \sqrt{\alpha}, \pm \sqrt{\beta}, \pm \sqrt{\gamma}).$$

The Jumelaire cubic of a point (λ, μ, ν) with respect to $P_1P_2P_3P_4$ is

$$\lambda x(\gamma y^2 - \beta z^2) + \mu y(\alpha z^2 - \gamma x^2) + \nu z(\beta x^2 - \alpha y^2) = 0.$$

So the net of Jumelaire cubics can be considered to be defined by the three degenerate cubics

$$x(\gamma y^2 - \beta z^2) = 0 \quad \ldots \quad \ldots \quad (1)$$

$$y(\alpha z^2 - \gamma x^2) = 0 \quad \ldots \quad \ldots \quad (2)$$

$$z(\beta x^2 - \alpha y^2) = 0 \quad \ldots \quad \ldots \quad (3)$$

Since S and O are isogonal conjugates, the polar conic of S with respect to (1) is a conic circumscribing the triangle formed by the lines $x = 0$, $y\sqrt{\gamma} \pm z\sqrt{\beta} = 0$ and has the line AO for a tangent at A. Hence it is a member of the pencil defined by the conic $\gamma z^2 - \beta y^2 = 0$ and the line-pair (AO, BC). Since O is the ortho-centre of the triangle, the lines AO and BC constitute a degenerate conic out-polar to the absolute and the line pair $\gamma y^2 - \beta z^2 = 0$ is also out-polar to the absolute. Hence all conics of the pencil defined by these two are out-polar to the absolute. Thus the polar conic of S with respect to (1), and similarly with respect to (2) and (3) is out-polar to the absolute. Hence every polar conic of S with respect to cubics of the Jumelaire net is out-polar to the absolute and so the polars of the absolute with respect to every cubic passes through S. The polar of the absolute with respect to the Jumelaire cubic of a point K necessarily passes through K and hence it is the line KS.

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* The ortho-centre of a $\triangle$ ABC may be defined as the centre of perspective of ABC and its polar triangle with respect to the absolute.

Among the cubics of the Jumelaire net there exists ∞^1 cubics admitting two more automorphic isogonal transformations of the first kind. Then, if the Jumelaire cubic Γ of K be one such the absolute has the same polar line with respect to all cubics of the syzygetic pencil defined by Γ. Hence the polar of the absolute with respect to the Hessian of Γ which passes through K must also pass through S. The condition that the polar line of the absolute in regard to the Hessian of Γ passes through S is found to be

$$(\beta'\gamma' + \alpha\alpha')\lambda(\gamma\mu^2 - \beta\nu^2) + (\gamma'\alpha' + \beta'\beta)\mu(\alpha\nu^2 - \gamma\lambda^2)$$
$$+ (\alpha'\beta' + \gamma\gamma')\nu(\beta\lambda^2 - \alpha\mu^2) = 0$$

where (λ, μ, ν) is the point K. This clearly shows that the point K lies on the Jumelaire cubic of the point S′ of co-ordinates

$$[(\beta'\gamma' + \alpha\alpha'), (\gamma'\alpha' + \beta\beta'), (\alpha'\beta' + \gamma\gamma')].$$

It can be easily seen that this point S′ is the isogonal conjugate of O with respect to the polar triangle A′B′C′ of ABC in regard to the absolute. Hence,

The only cubics among the net of Jumelaire cubics invariant for a given isogonal transformation of the first kind which admit of two more automorphic isogonal transformations of the first kind are the Jumelaire cubics of points on the Jumelaire cubic of S′, the isogonal conjugate of the ortho-cntre O with respect to the polar triangle A′B′C′ of ABC in regard to the absolute conic.[1]

GROUPS OF PLANE SYMMETRIES.

BY G. A. MILLER.

In the recent second edition of his *Theorie der Gruppen von endlicher Ordnung*, A. Speiser directed attention to the fact that the study of symmetries may have been the source of the earliest developments in higher mathematics of which we now have only indirect records, and that these developments may have preceded most of those relating to what we now regard as elementary mathematics which are embodied in the noted *Elements* of Euclid. He observed that the latter are naturally relatively more tedious and that it may reasonably be assumed that the earliest somewhat extensive mathematical developments were inspired by a sense of beauty and an interest in ornaments rather than by considerations relating to a logical foundation of mathematics. In particular, higher mathematics may have been studied about 1000 years earlier than has been commonly assumed hitherto, and it may have had its origin in Egypt about 1500 B C.

Since these supposed early developments in higher mathematics are intimately connected with the concept of group it may be of interest to note here some details relating to certain of the groups which are most evidently involved, without implying thereby that these details were explicitly formulated at such an early date. It seems possible that a more complete study of ancient ornaments as regards their geometric content will throw a new light on the early developments of mathematics and will exhibit, in particular, that the group concept played a prominent role in these developments. This would be in accord with a remark made by H. Poincare to the effect that the notion of group was one of the earliest mathematical notions to be assimilated by us, "and that consequently the notion of group existed prior to all others."* Euclid's *Elements* naturally exhibit the usual undue emphasis on the more recent

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* H. Poincare: *The Monist*, Vol. 9 (1898), p. 34.

acquisitions to the store of mathematical knowledge at the expense of the equally important older knowledge. This accounts reasonably for the fact that the notion of group failed to receive due recognition as long as mathematical work was so largely dominated by these *Elements.*

Proclus noted that Pythagoras knew that the space about a point in the plane can be completely covered by three regular hexagons, by four squares, or by six regular triangles. It is reasonable to assume that these facts were known at a much earlier date and that the infinite number of movements of the plane which transform the totality of these figures into themselves but interchange them as individuals were recognized very early, in spite of the fact that in Euclid's *Elements* the regular figures are studied as individuals, and hence only a finite number of movements are involved in their consideration. The fact that Euclid devoted so little attention even to this finite number of movements can perhaps be most satisfactorily explained if we assume that these ideas had then been so completely assimilated as to seem to require no special mention in a treatise.

It is well-known that a square is transformed into itself by four rotations around its centre and by four rotations around its lines of symmetry. These eight rotations constitute the well-known octic group, which may also be defined as the dihedral group of order 8. If we consider a plane filled with squares in such a manner that four of them meet always at the same point, the vertices constitute a lattice which is transformed into itself by ∞^2 translations of the plane, forming an abelian group H in which every operator besides the identity is of infinite order. This abelian group is an invariant sub-group of index 8 under the entire group G composed of all the plane movements and reflections which transform into themselves the totality of these squares but permute the individuals.

The group H is simply isomorphic with each of four regular substitution groups whose elements are the vertices of the squares, the centres of these squares, and the mid-points of their opposite sides respectively. The first two of these four sets of points are centres of rotation through angles of 90° and also through angles of 270°, while each

of the four sets is composed of centres of rotation through 180°. The only rotations of the plane around a fixed point which transform the totality of these squares into itself are evidently rotations through 0°, 90°, 180°, and 270° respectively. The rotations around the vertices and around the centres of the squares through 180° are squares of rotations through 90° around the same points, but the rotations around the mid-points of the sides of the squares through the same angle do not have this property.

In general, a translation followed by a rotation through a given angle α around a fixed point is equal to a rotation around a fixed point which may be found as follows: With the centre of the given rotation as an end-point construct a vector representing the translation in question, and from the parallelogram having this vector for one of its sides while its opposite side is bisected by the perpendicular to this vector through the centre of rotation and subtends an angle α at this centre. The end-points of this opposite side are the fixed points of the translation and rotation in question, taken in either of the two possible orders. In particular, such a translation and a rotation cannot be commutative. The fact that it is equal to a rotation around this fixed point through the angle α results directly from the facts that the distances between points were not affected by the given operations which were combined and that the points at a great distance from the fixed point must be moved through the angle α by the product of these two operators.

From the general considerations of the preceding paragraph, it results directly that the co-set of G with respect to H which involves a rotation through an angle of 90° is composed of rotations of period 4, representing rotations around the vertices and the centres of the given squares through 90°. Similar remarks obviously apply to the co-set which involves a rotation through 270°. On the contrary, the co-set which involves a rotation through 180° around a vertex of the given squares involves rotations through this angle not only around the vertices and the centres of the given squares but also around the mid-points of their opposite sides. What is perhaps most interesting in this connection is that we have here products of an infinite number of operators

each of an infinite order and a given operator of finite order, such that all of these products have for their common order the order of this finite operator. In view of the fact that these striking group theory properties are related to such a common object as a floor covered with square tiles, it seems that they deserve greater emphasis than they usually receive in our mathematical instruction.

The co-sets of G with respect to H which involve rotations around a line of symmetry of the given plane filled with squares are less striking. Each of them involves an infinite number of operators of order 2 as well as an infinite number of operator of infinite order, but no operator of any other order. The operators of order 2 result from the combination of an operator of order 2 which represents a rotation around a line of symmetry and the translations of H which are perpendicular to this line, while the operators of infinite order of such a co-set result from the combination of this operator of order 2 and any other translation of H. It may be noted here that these operators of order 2 are commutative with those translations of H which are parallel with the line of rotation in question but with no other operator of H.

While the groups relating to the plane filled with squares are quite interesting, those relating to the plane filled with regular hexagons are still more striking. The centres of these hexagons constitute a lattice in which the lines intersect at an angle of 60°. Hence the group G, formed by all the movements and reflections which transform these hexagons into each other, involves again an invariant abelian sub-group H, composed of the translations represented by the given lattice, whose unit distances are twice the apothems of the said regular hexagons. The index of H_1 under G_1 is 12 and the corresponding quotient group is dihedral. The co-sets corresponding to rotations through 60° or through —60° are composed of operators of order 6 representing rotations around the centres of the hexagons only, while the co-sets corresponding to rotations through 120° or through —120° are composed of operators of order 3 representing rotations not only around the centres of the given hexagons but also around their vertices. The co-set

corresponding to rotations through 180° is composed of operators o order 2 representing rotations around the centres of the given hexagons and also around the mid-points of their sides.

Each of the remaining six co-sets of G_1 with respect of H_1 involves an infinite number of operators of order 2 representing the products of a fixed rotation around a line of symmetry and a translation at right angles to this line. All the other operators in each of these six co-sets are of infinite order and represent the ∞^2 products of an operator of order 2 in such a co-set and any operator of H_1 which does not represent a translation which is perpendicular to the lines of symmetry corresponding to the co-set. These considerations relating to the groups of movements and reflections of the plane filled with regular hexagons clearly apply also to the plane filled with regular triangles since such a plane is formed by connecting the centres of these hexagons. These groups are of special interest not only because they are involved in the theory of ornaments but also because they may possibly be the first group to be seriously considered by the human race.

QUADRICS OF REVOLUTION THROUGH A GIVEN CONIC.

By A. Narasinga Rao.

1. Introduction.

From the view-point of Projective Geometry, the quadrics considered in this paper are those which pass through one conic and have double contact with another.

When a quadric becomes singular tangentially, it stands for a definite plane and a definite conic in that plane. Hence two conics in space determine two other covariant conics, all the four being the singular members of the same tangential pencil. Such a system of *focal conics* have the property that any plane touching two of the conics touches all the four. From this follows the characteristic property of a focal system, *viz.*:

Any cone having its vertex on one conic and passing through another has double-contact with each of the other two members of the focal system ... ... (1)

a property which will be found useful later.* When one of the conics is taken to be the Absolute of a Euclidean metric, the conics become the focal conics of a confocal system, and the quadrics having double-contact with it are "quadrics of revolution."

The properties of such quadrics of revolution discussed in this paper are all more or less well known. The distinctive feature of the present treatment is the orientation given to it by the concept that each quadric of revolution through a given conic can be obtained by revolving the conic about some line in space, and the consequent representation of quadrics of the system by the straight lines of a (degenerate) congruence.

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* For a more detailed treatment of this subject reference may be made to Baker: ***Principles of Geometry***, Vol. III. page 95.

It is remarkable how many of the essential properties of the system may be deduced by elementary reasoning and without the use of co-ordinates.

2. The axes of revolution of quadrics through a given conic Γ.

When a conic Γ rotates about a line l in space, the resulting surface of revolution is, in general, of degree four; for a section by any plane perpendicular to l consists of a pair of circles, namely those traced out by the two points P, Q in which the plane cuts Γ. If, however, these two points be equidistant from l, they trace the same circle, and should this happen for every plane perpendicular to l, the resulting surface will be a quadric. This condition is clearly both necessary and sufficient.

Now, the planes perpendicular to l determine a system of parallel chords of Γ, and the planes bisecting these at right angles are all parallel and will have a common line l only if they all coincide, *i.e.* if the chords are parallel to a principal axis of Γ. Hence

the congruence of lines l about which a conic Γ may be rotated, so that the resulting surface may be a quadric, breaks up into two plane systems, namely, those lying in either of the principal planes of Γ. ... ... (2)

It will be seen later that the distinction between the two systems is closely connected with that which is associated with the words 'prolate' and 'oblate.'

3. Special members of the Quadric System.

Let l be any line in either of the principal planes of Γ and Q_l the corresponding quadric of revolution. We next proceed to consider special members of the quadric system corresponding to particular positions of l.

If Q_l be a cone, Γ is one of its plane sections and by a well-known theorem in geometrical conics the product of the perpendiculars from the extremeties A, A$'$ of the principal axis of Γ coplanar with l is constant and equal to the square of the other axis. Hence the envelope of the line l is a conic having these two vertices of Γ for foci and the corresponding foci for vertices. There are two such conics one in each

of the principal planes. These are the focal conics of Γ, and all the three conics are members of a tangential pencil to which belongs also the absolute conic, *i.e.* members of a certain confocal system of quadrics.

To determine the vertex P of the cone Q_l, we notice that since it is one of revolution, and APA′ is a section through the axis, PA and PA′ should be equally inclined to l. But l is a tangent to the focal conic and the join of its point of contact to the foci A, A′ are equally inclined to l. Hence the point of contact is the vertex of Q_l, and we have the well-known result that

> the focal conics are the loci of the vertices of cones of revolution through Γ.*

The only other types of special quadrics under the projective group are a pair of planes and a repeated plane. The former is impossible unless Γ is imaginary and consists of a pair of straight lines, and the latter corresponds to rotation about the unique line common to the two principal planes.

4. Two quadrics of revolution may be called "similar" if they have the same asymptotic angle. Now the asymptotic cone is also one of revolution and its semi-vertical angle is the angle between l and either of the asymptotes of Γ. Hence

> the quadrics corresponding to a series of parallel lines l, are all similar and similarly situated. ... ... (3)

Among these there will be two cones corresponding to the two positions in which l is a tangent to the focal conic in its plane. If we take the reflection of l in any of the three principal planes of Γ and take the series of lines parallel to it we shall obtain another system of similar and similarly situated quadrics with the same asymptotic angle.

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* More generally if the enveloping cone from a point to a quadric of a confocal system be one of revolution, its vertex lies on one of the focal conics. Salmon : *Geometry of three dimensions*, revised by Rogers, Vol. I, p. 186-187.

5. We shall now prove that:

the principal foci of any quadric Q_l are the intersections of its axis l with the focal conic co-planar with it. ... (4)

Let S be a point common to l and a focal conic. Then from the fundamental property of focal conics, a cone with vertex S and passing through the absolute conic (which is also a member of the focal system) has double contact with Γ. But such a cone is also a point sphere with its centre at S on the axis of Q_l and hence cuts Q_l in a pair of circles. The intersections of these circles with Γ are therefore the intersections of the cone, or point sphere, with Γ. But we have seen above that the point sphere touches Γ. Therefore the two circles coincide, *i. e.*, the point sphere centre S has ring contact with the quadric S, is therefore a principal focus.

As immediate corollories from the above we deduce that the centre of Q_l is the mid-point of the chord l of the focal conic; and that the locus of the centres of similar and similarly situated quadrics of the system is a line through the centre of Γ.

6. Metrically noteworthy members of the system.

When one of the foci is at infinity we have a paraboloid Hence

the paraboloids of revolution through Γ correspond to lines parallel to one or other of the asymptotes of a focal conic. When l actually coincides with an asymptote, the quadric becomes a right circular cylinder ... (5)

since it is a cone with its vertex at infinity.

The plane of Γ and its two principal planes of symmetry are mutually at right angles, and the group generated by reflections in these planes is of order 8. Under this group of reflections a quadric Q_l goes over into any one of a set of four, since reflection in the principal plane in which l lies carries the quadric Q_l into itself These are the quadrics which are equivalent under the metrical group of movements and reflections and it is easy to prove that there are no others. Hence:

there are 4 congruent quadrics of revolution through Γ. The lines l of such a set of four form a rhombus whose diagonals are the lines of symmetry of Γ.

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NOTES AND QUESTIONS.

ASSOCIATE EDITORS:

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Notes and Questions.

(2 - 1) Correspondence between two real variables.

1. In the February issue of this *Journal*, I discussed the correspondence between two complex variables z and w, connected by the relation

$$z^2(a'w-a)+z(b'w-b)+(c'w-c)=0 \qquad \dots \quad (1)$$

or

$$w=\frac{az^2+bz+c}{a'z^2+b'z+c'}\equiv R(z) \text{ (say)}. \dots \qquad \dots \quad (2)$$

In the present note, the variables z and w as well as the co-efficients in the form (1) will be further restricted to be real numbers. It will be shown that when the variables w and z are subjected to *real* homographic transformations, the correspondence (1) falls into one or other of two classes, *viz.*, those defined by

$$w=\tfrac{1}{2}\left(z+\frac{1}{z}\right)\text{ ; and } w=\tfrac{1}{2}\left(z-\frac{1}{z}\right).$$

Since the co-efficients in (1) are real, the roots of the quadratic in z corresponding to any given real value of w, must be either both real or both imaginary. Since we do not consider unreal values, this means that w cannot take all values and leads to problems of maxima and minima which are considered in the last paragraph.

The two variables may be represented geometrically by points on two real lines, or generally on two real curves of genus zero. A geometrical construction may then be given which associates each point w on one curve with a pair of points z on the other, and with each point z on the second curve a definite point w on the first. A good example of such a geometric representation is the (2 - 1) correspondence of points z on a circle and the points w where the corresponding tangent to the circle meets a given line.

2. *The two canonical forms.*

The two companion values z_1 and z_2 which correspond to the same value of w are roots of quadratics of the pencil

$$(az^2 + bz + c) - w(a'z^2 + b'z + c') = 0 \qquad \dots \quad (3)$$

and hence belong to an involution whose double points correspond to the perfect squares in the pencil. By means of a translation and a similarity transformation, *i.e.*, by a homography of the type $z' = az + b$, where a and b are real, the involution may be reduced to the form $z'_1 z'_2 = \pm 1$ according as the double points are real or imaginary.

Again it is clear from (3) that the correspondence between w and $z_1 + z_2$ (or $z'_1 + z'_2$) is (1,1) so that we may write

$$H(w) = \tfrac{1}{2}(z'_1 + z'_2) = \tfrac{1}{2}\left(z'_1 \pm \frac{1}{z'_1}\right)$$

where H is a homographic operator with real co-efficients. Hence:

Every real (2-1) correspondence can be reduced to either of the forms

$$w = z + \frac{1}{z} \text{ or } w = z - \frac{1}{z}$$

by real homographic transformations applied to w and z.

3. *Reduction to canonical form*

The actual reduction to the canonical form may be effected as follows: Write

$$w = R(z) = \frac{az^2 + bz + c}{a'z^2 + b'z + c'} = A + \frac{1}{Bz + C +} \frac{1}{Dz + E}.$$

If B and D are of the same sign, we take

$$H_1(z) = z' = \sqrt{BD}\,(z + E/D)$$

$$z'' = Q(z') = \tfrac{1}{2}\left(z' + \frac{1}{z'}\right)$$

Then

$$R(z) = w = A + \frac{1}{z'\sqrt{B/D} + C - BE/D +} \frac{1}{z'\sqrt{D/B}}$$

$$= A + \frac{1}{2z''\sqrt{B/D} + C - BE/D} \equiv H_2(z'').$$

Hence

$$R(z) = H_2 . Q . H_1(z).$$

where H_1 and H_2 are definite homographics,

When B and D are opposite in sign, we write

$$z' = H_1(z) = \sqrt{-BD}\,(z + E/D); \quad Q(z') = \tfrac{1}{2}\left(z' - \frac{1}{z'}\right)$$

$$H_2(z) = A + \frac{1}{2z\sqrt{-B/D} + (C - BE/D)}.$$

and have

$$R(z) = H_2 . Q . H_1(z)$$

the correspondence being now of the second type.

4. *Applications to maxima and minima.*

When the correspondence reduces to $w' = z' - \frac{1}{z'}$, there exist values of w' corresponding to any z' since the defining equation in z' has always real roots; in the other case, where the canonical form is $w = z' + \frac{1}{z'}$, w' cannot take values between $+1$ and -1. The maxima and minima values of the original function $R(z)$ may now be obtained by finding the values of w corresponding to $w = \pm 1$.

We may, however, find the turning values directly as in the following examples.

Example (i). To find the range of values of :

$$y = \frac{18x^2 - 27x + 3}{9x^2 - 15x + 4} = R(x)$$

when x takes all real values.

Convert $R(x)$ into a continued fraction,

$$\frac{18x^2 - 27x + 3}{9x^2 - 15x + 4} = 2 + \frac{1}{3x +}\;\frac{4}{3x - 5}$$

Now

$$3x - 5 + \frac{4}{3x - 5} = \left(\sqrt{3x - 5} - \frac{2}{\sqrt{3x - 5}}\right)^2 + 4,$$

if $3x - 5$ is positive.

$$= -\left(\sqrt{5 - 3x} - \frac{2}{\sqrt{5 - 3x}}\right)^2 - 4,$$

if $3x - 5$ is negative.

So $\left(3x-5+\frac{4}{3x-5}\right)$ takes values above 4 and also values below -4 including the extreme values.

$\therefore \quad 3x+\frac{4}{3x-5}$ takes values outside the range $+9$ and $+1$.

$\therefore \quad \frac{1}{3x+\frac{4}{3x-5}}$ takes values outside $\frac{1}{9}$ and 1.

$\therefore \quad y=2+\frac{1}{3x+\frac{4}{3x-5}}$ takes values outside $2\frac{1}{9}$ and 3.

Therefore 3 is a minimum value and $2\frac{1}{9}$ is a maximum.

Example (ii). $\quad y=\frac{4\sin^2\theta+10\sin\theta-25}{4\sin^2\theta+\ 4\sin\theta-16}$

where θ takes all real values. It is required to find the range of values of y.

Put $\quad \sin\theta=x$.

Then $y=\frac{4x^2+10x-25}{4x^2+\ 4x-16}$ where x takes only values between $+1$ and -1.

Convert $\frac{4x^2+10x-25}{4x^2+\ 4x-16}$ into a continued fraction;

$$\frac{4x^2+10x-25}{4x^2+\ 4x-16}=1+\frac{3}{2x+5-\frac{1}{2x-3}}$$

As x changes from -1 to $+1$, $2x-3$ changes from -5 to $+1$, *i.e.*, from -5 to zero and zero to 1. Now it may be remarked that $\frac{1}{2}\left(z-\frac{1}{z}\right)$ increases whether z moves along the x axis from $-\infty$ to zero, or from 0 to $+\infty$. As $(2x-3)$ changes from -5 to 0, $2x-3-\frac{1}{2x-3}$ increases from $-4\frac{4}{5}$ to $+\infty$ and as $2x-3$ in-

creases from 0 to + 1, $2x-3-\frac{1}{2x-3}$ increases from $-\infty$ to 0. Therefore as x changes from -1 to $+1$, $\left(2x-3-\frac{1}{2x-3}\right)$ and therefore also y takes all real values from $-\infty$ to $+\infty$.

S. PURUSHOTHAM.

Note on a Theorem of Pappus.

§ 1. The theorem of this note is referred to by Pappus, a Greek Mathematician, who lived about 300 A.D., as an ancient proposition and is proved by him with the help of several auxiliary Lemmas. A particular case of this theorem occurs as Proposition 6 in the Book of Lemmas due to Archimedes.* It is the object of this note to give a new proof of this theorem based on the following well-known equation for the radii of circles touching three circles of radii a, b, c having mutual contact:—

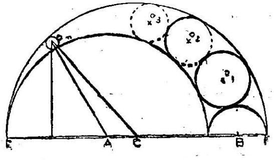
$$x^2\left(\Sigma\frac{1}{a^2}-2\Sigma\frac{1}{ab}\right)-2x\Sigma\frac{1}{a}+1=0. \quad \ldots \quad (1)$$

This equation has been proved in the columns of our Journal.† An obvious deduction from (1) is that if r, r' be the radii of the two circles which touch the given circle, then

$$\frac{1}{r}+\frac{1}{r'}=2\sum\frac{1}{a} \quad \ldots \quad \ldots \quad (2)$$

the radii being taken with their proper sign.

§ 2. PAPPUS' THEOREM. A, B, C are the centres of three semicircles of radii a, b, and c each in contact with the other two as in



the figure. Circles with centres O_1, O_2, O_3 ... O_n are drawn so that

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* *Vide* T. L. Heath: *The Works of Archimedes*, pp. 307-8.

† *Vide* J.I.M.S., Vol. XII, p. 237 and Vol. XIII, pp. 22-23.

the first of them touches all the three semi-circles, the second touches the first and the two semi-circles touching at E and so on as in the diagram; then the perpendicular from the centre O_n of the nth circle on the diameter EF is n times the diameter of that circle.

In particular, the perpendicular from O_1 to EF is equal to the diameter of the first circle which touches all the three semi-circles.

For purposes of (1) we have to consider the radius c of the circle on EF as diameter as negative, so that the numerical value of the radius is $-c$.

The condition of mutual contact of the three semi-circles is given by

$$a + b + c = 0,$$

i.e.,

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = 0.$$

With this condition, the equation of (1) reduces to

$$x^2\left(\Sigma\frac{1}{a^2}\right) - 2x\,\Sigma\frac{1}{a} + 1 = 0$$

or

$$x^2\left(\Sigma\frac{1}{a}\right)^2 - 2x\,\Sigma\frac{1}{a} + 1 = 0$$

∴

$$x = \frac{1}{\Sigma\frac{1}{a}} \text{ or } \frac{1}{x} = \Sigma\frac{1}{a}.$$

Denote by $r_1, r_2, \ldots r_n$ the radii of the n circles in question; the repeated application of (2) gives

$$\frac{1}{r_1} = \frac{1}{b} + \frac{1}{a} + \frac{1}{c}$$

$$\frac{1}{r_2} + \frac{1}{b} = 2\left(\frac{1}{r_1} + \frac{1}{a} + \frac{1}{c}\right)$$

$$\frac{1}{r_3} + \frac{1}{r_1} = 2\left(\frac{1}{r_2} + \frac{1}{a} + \frac{1}{c}\right)$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$\frac{1}{r_n} + \frac{1}{r_{n-2}} = 2\left(\frac{1}{r_{n-1}} + \frac{1}{a} + \frac{1}{c}\right).$$

Adding these relations and removing the common terms from both sides, we get

$$\frac{1}{r_n} - \frac{1}{r_{n-1}} = (2n-1)\left(\frac{1}{a}+\frac{1}{c}\right). \quad \dots \quad \dots (3)$$

Replacing n by $n-1$, $n-2$, ... etc, in (3), we get

$$\frac{1}{r_{n-1}} - \frac{1}{r_{n-2}} = (2n-3)\left(\frac{1}{a}+\frac{1}{c}\right)$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\frac{1}{r_2} - \frac{1}{r_1} = 3\left(\frac{1}{a}+\frac{1}{c}\right)$$

$$\frac{1}{r_1} = \frac{1}{b} + \left(\frac{1}{a}+\frac{1}{c}\right)$$

whence

$$\frac{1}{r_n} = \frac{1}{b} + \left(\frac{1}{a}+\frac{1}{c}\right)\left(1+3+\dots+\overline{2n-1}\right)$$

$$= \frac{1}{b} + \left(\frac{1}{a}+\frac{1}{c}\right)n^2 \quad \dots \quad \dots \quad \dots (4)$$

an interesting result, which, since $a+b+c=0$, may be written,

$$(r_n - b)\,ac = n^2b^2r_n.$$

Now, let C be the mid-point of EF; then the sides of the triangle O_nAC are $-c-a$, $-c-r_n$ and $a+r_n$ and the perpendicular from O_n on EF

$$= \frac{2\,\Delta\, O_n AC}{AC} = \frac{2\sqrt{ac\,(r_n-b)\,r_n}}{b}$$

$$= 2\,nr_n$$

$= n$ times the diameter of the nth circle.

It is interesting to note (i) that the ex-radius of the triangle O_nAC opposite to the vertex A is n.AC, and (ii) that the centres O_1, O_2, ... O_n lie on an ellipse whose foci are A, C and vertices E, B.

A. A. KRISHNASWAMI AYYANGAR.

Solutions.

Question 629.

(S. RAMANUJAN) :—Prove that

$$\frac{\frac{1}{2}+\sum_{n=1}^{\infty} e^{-\pi n^2 x}\cos\left(\pi n^2\sqrt{1-x^2}\right)}{\sum_{n=1}^{\infty} e^{-\pi n^2 x}\sin\left(\pi n^2\sqrt{1-x^2}\right)}=\frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{1-x}}$$

and deduce the following :—

(a) $$\frac{\frac{1}{2}+\sum_{1}^{\infty} e^{-\pi n^2}}{\frac{1}{2}+\sum_{1}^{\infty} e^{-5\pi n^2}}=\sqrt{5\sqrt{5}-10}$$

(b) $$\sum_{1}^{\infty} e^{-\pi n^2}\left(\pi n^2-\tfrac{1}{4}\right)=\tfrac{1}{8}.$$

Remarks by S. D. Chowla.

On p. 25, *J. I. M. S.*, Feb. 1916, the sums of the infinite series

$$\pi\sum_{1}^{\infty} e^{-\pi n^2}.\,n^2 \quad\text{and}\quad -\tfrac{1}{4}\sum_{1}^{\infty} e^{-\pi n^2}$$

are incorrectly given as $\frac{1}{4}$ and $-\frac{1}{8}$ respectively. The following method gives the required sums in terms of Γ-functions.

We know from the theory of elliptic functions that

$$1+2q+2q^4+2q^9+\ldots=\sqrt{\frac{2K}{\pi}} \qquad \ldots\ (1)$$

where $q=e^{-\pi K'/K}$.

Also since $$EK'+KE'-KK'=\frac{\pi}{2} \qquad \ldots \quad \ldots\ (2)$$

it follows that $$\frac{dq}{dk}=\frac{\pi^2 q}{2kk'^2K^2} \qquad \ldots \quad \ldots\ (3)$$

$\therefore$ (1) gives

$$\frac{2}{q}\sum_{1}^{\infty} n^2 q^{n^2} = \frac{1}{\sqrt{2\pi}}\,\frac{1}{\sqrt{K}}\,\frac{dK}{dq} = \frac{1}{\sqrt{2\pi K}}\left(\frac{dK}{dk} \div \frac{dq}{dk}\right)$$

$$= \frac{1}{\sqrt{2\pi\,K}}\;\frac{E - k'^2 K}{kk'^2}\;\frac{2kk'^2 K^2}{\pi^2 q} \qquad \text{from (3)}$$

$$\therefore \qquad \sum_{1}^{\infty} n^2 q^{n^2} = \frac{1}{\sqrt{2}\,\pi^{\frac{5}{2}}}\,(E - k'K^2)\,K^{\frac{3}{2}}. \qquad \ldots \quad (4)$$

Again if $k = \frac{1}{\sqrt{2}}$, it is known that

$$K = K' = \frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}}.$$

Hence from (2) we get

$$E = \frac{\pi \,.\, 4\sqrt{\pi}}{4\,\Gamma^2(\frac{1}{4})} + \frac{\Gamma^2(\frac{1}{4})}{8\sqrt{\pi}},\ \left(k = \frac{1}{\sqrt{2}}\right).$$

Hence from (4)

$$\sum_{1}^{\infty} n^2 e^{-\pi n^2} = \frac{1}{\sqrt{2}\,\pi^{\frac{5}{2}}}\,\frac{\pi}{4}\,\sqrt{K}\,\left(k = \frac{1}{\sqrt{2}}\right)$$

$$\therefore \quad 8\pi \sum_{1}^{\infty} n^2 e^{-\pi n^2} = \sqrt{\frac{2K}{\pi}} = \sqrt{\frac{\Gamma^2(\frac{1}{4})}{2\pi^{\frac{3}{2}}}} = \frac{\Gamma(\frac{1}{4})}{\sqrt{2}\,\pi^{\frac{3}{4}}} \qquad \ldots \quad (5)$$

and from (1)

$$\sum_{1}^{\infty} e^{-n^2\pi} = \frac{\sqrt{\frac{2K}{\pi}} - 1}{2} = \frac{\frac{\Gamma(\frac{1}{4})}{\sqrt{2}\pi^{\frac{3}{4}}} - 1}{2} \qquad \ldots \quad (6)$$

From the last two results, we get

$$\sum_{1}^{\infty} \left\{ e^{-n^2\pi}\left(\pi n^2 - \tfrac{1}{4}\right) \right\} = \tfrac{1}{8}$$

which is (*b*).

Question 1316.

(R GOPALASWAMI):—Defining the 'inconic of P with respect to a triangle ABC' as the conic touching the sides of ABC at the points where they are cut by AP, BP, CP respectively, prove that:

(i) the inconic of the ortho-centre has its centre at the Lemoine point and its axes parallel to the asymptotes of the rectangular hyperbola passing through the vertices and the Gergonne point of the pedal triangle.

(ii) the inconic of the isotomic conjugate of the circum-centre has double contact with the nine points-circle.

Solution by M. V. Seshadri.

(i) Let D E, F be the feet of the perpendiculars from A, B, C, O the ortho-centre of ABC and G the Gergonne point of DEF. Then EF is the chord of contact of the tangents from A to the inconic of O. The line through A bisecting EF passes through the centre of this conic. But EF is antiparallel to BC. The line through A bisecting EF is the symmedian through A. Similarly the symmedians through B and C contain the centre which is hence the Lemoine point of the triangle.

Next to find the directions of the axes: the normals at D, E, F to the conic are OD, OE, and OF and they are concurrent at O. Hence a rectangular hyperbola passes through DEFO and the centre of the conic (the Lemoine point of ABC) and this has the asymptotes parallel to the axes of the conic. Now we shall prove that this rectangular hyperbola which passes through the vertices and the incentre of the pedal triangle passes also through the Gergonne point.

Now the equation of a rectangular hyperbola through the vertices of the reference triangle is

$$fyz + gzx + hxy = 0$$

where we have

$$f \cot \alpha + g \cot \beta + h \cot \gamma = 0$$

and α, β, γ are the angles of the triangle. Also since the hyperbola is to pass through the incentre whose areal co-ordinates are proportional to a, b, c

we have $$\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0 \qquad \ldots \qquad \ldots \qquad (2)$$

where a, b, c are the sides of the triangle.

From the two we get

$$\frac{f}{\frac{\cot\beta}{c}-\frac{\cot\gamma}{b}}=\frac{g}{\frac{\cot\gamma}{a}-\frac{\cot\alpha}{c}}=\frac{h}{\frac{\cot\alpha}{b}-\frac{\cot\beta}{a}}$$

i.e. f, g, h are proportional to $a\,(c\cot\gamma - b\cot\beta)$ etc.

But $a\,(c\cot\gamma - b\cot\beta)$

$$=\frac{a\,[c\,(a^2+b^2-c^2)-b\,(c^2+a^2-b^2)]}{4\,\Delta}=\frac{a\,(b+c-a)\,(b+c+a)\,(b-c)}{4\,\Delta}$$

$\therefore$ f, g, and h are proportional to

$$a\,(s-a)\,(b-c),\ b\,(s-b)\,(c-a) \text{ and } c\,(s-c)\,(a-b).$$

Now the areal co-ordinates of the Gergonne point are respectively proportional to

$$\frac{1}{s-a},\ \frac{1}{s-b},\ \frac{1}{s-c}$$

If G is to be on the hyperbola $\Sigma f\,(s-a)=0$

i.e. $\Sigma\, a\,(s-a)^2\,(b-c)=\Sigma\, a\,(b-c)\,(s^2-2sa+a^2)$

must be zero, which is easily verified.

Hence the rectangular hyperbola through the vertices and incentre of any triangle passes through its Gergonne point. So the rectangular hyperbola through DEF and O passes through G. Thus the axes of the inconic of O are parallel to the asymptotes of this rectangular hyperbola which passes through the vertices and Gergonne point of the pedal triangle.

(ii) For a solution to the second part of the question, *vide J.I.M.S.*, Vol. XVII, No. 4 (August 1927), p. 60.

Question 1419.

(S. D. CHOWLA):—Prove that

(i) $e^z = (1 + \tanh z)(1 + \tanh z_1)(1 + \tanh z_2) \ldots$ *ad. inf.*

where $e^{4z} - 2e^{2z}(2e^{2z_1} + 1) + 1 = 0,$

and z_1, z_2 are connected by a similar relation, and so on.

(ii) $$\operatorname{cosec}\theta = \left(1 + \frac{1}{\sec^2\theta + \tan^2\theta}\right) \times \left(1 + \frac{1}{\sec^2\theta_1 + \tan^2\theta_1}\right) \ldots \text{ ad. inf.}$$

where $\tan\theta_1 = 2\tan\theta\sec\theta$

and in general $\tan\theta_n = 2\sec\theta_{n-1}\tan\theta_{n-1}$.

Solution by G. V. Krishnaswami.

(i) There is a slight mistake in the relation between z and z_1 as given in the question; it ought to be

$$e^{4z} - 2e^{2z}(2e^{2z_1} - 1) + 1 = 0$$

$$\therefore \quad e^{4z} + 2e^{2z} + 1 = 4e^{2(z+z_1)}$$

$$\therefore \quad e^{2z} + 1 = 2e^{z+z_1}$$

$$\therefore \quad e^{z_1} = \frac{e^z + e^{-z}}{2} = \frac{e^z}{1 + \tanh z}.$$

i.e. $e^z = e^{z_1}(1 + \tanh z)$:

Hence the result.

(ii) $\tan\theta_1 = 2\tan\theta\sec\theta = 2\sin\theta/\cos^2\theta$

$$\therefore \quad \operatorname{cosec}\theta_1 = \sqrt{\cos^4\theta + 4\sin^2\theta}\,/\,2\sin\theta = \frac{1+\sin^2\theta}{2\sin\theta}$$

$$\therefore \quad \operatorname{cosec}\theta_1 = \operatorname{cosec}\theta\left(\frac{1+\sin^2\theta}{2}\right)$$

$$= \operatorname{cosec}\theta \Big/ \left(1 + \frac{1}{\sec^2\theta + \tan^2\theta}\right)$$

$$\therefore \quad \operatorname{cosec}\theta = \left(1 + \frac{1}{\sec^2\theta + \tan^2\theta}\right)\operatorname{cosec}\theta_1$$

Hence the result.

Question 1421.

(SELECTED);—A straight line PQRS intersects two confocal ellipses at the points P, Q, R, S in order. If the tangents at P and R are perpendicular, prove that the tangents at Q and S are also perpendicular.

(1) *Geometrical Solution by S. Audinarayanan and G. V. Krishnaswami.*

It is a well-known property of co-axial circles that if L be one of the limiting points of two circles, and LA, LB are two radii vectors at right angles to each other, and terminating in these circles, the locus of intersection of tangents at A and B is a circle co-axial with the two given circles.

So if E be the point of intersection of the tangents, and the other two tangents to the two circles be EC, ED, then it is evident that LC, LD are also at right angles. Now reciprocate with respect to L. The two circles reciprocate into two confocal conics, E reciprocates into a straight line PQRS, lines EA, EB, EC, ED, reciprocating into the points P, R, Q and S respectively. Since LA, LB are at right angles, the tangents at P and R are at right angles, and hence also the tangents at Q and S are at right angles.

The envelope of such lines will be seen to be a confocal conic.

(2) *Analytical Solution by N. Durairajan and S. Rajanarayanan.*

Let the equations of the confocal ellipses be

$$L \equiv \frac{x^2}{a^2+L} + \frac{y^2}{b^2+L} - 1 = 0 \qquad \text{... (1)}$$

$$M \equiv \frac{x^2}{a^2+M} + \frac{y^2}{b^2+M} - 1 = 0. \qquad \text{... (2)}$$

Let the tangent at $P\,(x_1\, y_1)$ on the former and $R\,(x_2\, y_2)$ on the latter be perpendicular.

Write
$$L_{rs} = \frac{x_r\, x_s}{a^2+L} + \frac{y_r\, y_s}{b^2+L} - 1 \qquad \text{... (3)}$$

$$M_{rs} = \frac{x_r\, x_s}{a^2+M} + \frac{y_r\, y^s}{b^2+M} - 1. \qquad \text{... (4)}$$

Then the following conditions hold

$$L_{11} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

$$M_{22} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

$$\frac{x_1 x_2}{(a^2 + L)(a^2 + M)} + \frac{y_1 y_2}{(b^2 + L)(b^2 + M)} = 0 \quad \ldots \quad (7)$$

$$\therefore \quad L_{12} = M_{12} \quad \ldots \quad \ldots \quad (8)$$

Let the other intersections of PR with the conics be

$$S: \quad \left(\frac{x_1 + \lambda_1 x_2}{1 + \lambda_1}, \frac{y_1 + \lambda_1 y_2}{1 + \lambda_1} \right) \text{ lying on (1)}$$

and $$Q: \quad \left(\frac{x_1 + \lambda_2 x_2}{1 + \lambda_2}, \frac{y_1 + \lambda_2 y_2}{1 + \lambda_2} \right) \text{ lying on (2).}$$

Substituting, we get the following values for λ_1 and λ_2;

$$\lambda_1 = -2 \frac{L_{12}}{L_{22}}, \quad \lambda_2 = -\tfrac{1}{2} \frac{M_{11}}{M_{12}} \quad \ldots \quad (9)$$

The condition that the tangents at S and Q are perpendicular is

$$\frac{(x_1 + \lambda_1 x_2)}{(1 + \lambda_2)(1 + \lambda_1)} \cdot \frac{(x_1 + \lambda_2 x_2)}{(a^2 + L)(a^2 + M)} + \ldots = 0 \ldots \quad (11)$$

This expression when simplified becomes

$$L_{11} + (\lambda_1 + \lambda_2) L_{12} + \lambda_1 \lambda_2 L_{22}$$

$$= M_{11} + (\lambda_1 + \lambda_2) M_{12} + \lambda_1 \lambda_2 M_{22} \quad \ldots \quad (12)$$

$$\therefore \quad \lambda_1 \lambda_2 . L_{22} = M_{11} \quad \ldots \quad \ldots \quad (13)$$

which is true from (9)

Hence the theorem is proved.

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1506. (M. BHIMASENA RAO):—S, S′ are a pair of isogonal conjugate points with respect to a triangle ABC; P, P′ are a pair of harmonic conjugates with respect to S, S′, and Q, Q′ are isogonal conjugate points on a variable line through P. Show that the envelope of the radical axes of the pedal circles of S and Q is a conic whose centre is midway between the orthocentre of ABC and the middle point of PP′, and that the same conic is obtained if we consider P′ instead of P.

1507. (N. DURAIRAJAN):—Given four points in a plane, show that the conics which have their centres at any one of the four points and have the triangle formed by the other three for a self-polar triangle are similar conics.

1508. (N. DURAIRAJAN):—Show that the condition that four points A, B, C, D may be co-normal points on some conic through them is

$$\sum_{1}^{3} a^{3}\ (R_2^{\,2} - R_3^{\,2})\ (RR_1 + R_2R_3)^2 = 0,$$

where a, b, c are the lengths of the sides of the triangle ABC and R, R_1, R_2, R_3 are the circum-radii of the triangles ABC, DBC, DCA, DAB, respectively.

1509. (S. D. CHOWLA):—If $-\frac{\pi}{7} < x < \frac{\pi}{7}$, prove that

$$\sum_{1}^{\infty} \frac{a_n \cos nx}{n} = 0,$$

where
$$a_1 = a_6 = a_9 = a_{10} = a_{11} = a_{12} = +1,$$
$$a_2 = a_3 = a_4 = a_5 = a_8 = a_{13} = -1.$$
$$a_7 = a_{14} = 0,\ a_n = a_{n+14}.$$

1510. (B. B. BAGI) :—The points $A_1, A_2, \dots A_n$ represent the complex numbers $\alpha_1, \alpha_2 \dots \alpha_n$, and $B_1, B_2, \dots B_n$, the $n-1$ roots of the equation

$$\frac{\lambda_1}{x-\alpha_1} + \frac{\lambda_2}{x-\alpha_2} + \dots + \frac{\lambda_n}{x-\alpha_n} = 0,$$

where $\lambda_1, \lambda_2, \dots \lambda_n$ are real. Show that

$$\frac{A_1B_1 . A_1B_2 \dots A_1B_{n-1}}{A_1A_2 . A_1A_3 \dots A_1A_n} + \frac{A_2B_1 . A_2B_2 \dots A_2B_{n-1}}{A_2A_1 . A_2A_3 \dots A_2A_n}$$
$$+ \dots \text{ to } n-1 \text{ terms} = 1.$$

1511. (S. N. KUMARASWAMI) :—In a triangle ABC, M is the middle point of BC, K the point of contact of the incircle with BC, AD the altitude, I the incentre, O the orthocentre and Q the middle point of AO. Show geometrically that

$$MK . KD + IK . QD = Rr.$$

1512. (A. A. KRISHNASWAMI AYYANGAR) :—If the sides of a triangle are denoted by the equation $f(x, y) = 0$, in rectangular co-ordinates, show that the radical axis of the circum-circle and the nine-points circle of the triangle is given by

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

1513. (Prof. K. J. SANJANA) :—Prove that x being a positive integer, the solution in rational numbers of the equation

$$M^2 - (2x+1) N^2 = \pm (x^2 - 1)$$

will serve to factorise numbers of the form $N_1^4 \pm N_1^2 N_2^2 + N_2^4$, and find the law of the series formed by the successive values of x.

1514. (Prof. K J. SANJANA) :—Prove that the solution in rational nnmbers (integral or fractional) of the equation

$$M^2 - 2x N^2 = \pm (x^2 - 1)$$

will serve to factorise numbers of the form $N_1^4 + N_2^4$; show further that x, taken to be a positive integer, is always of the form $4p^2 + 1$.

Pamphlets.

University of California Publications in Mathematics—(contd.).

4 A Complete Set of Postulates for the Logic of Classes expressed in terms of the Operat[illegible]n "Exception," and a proof of the independence of a set o[illegible] Postulates due to Del Ré., by Bernstein. Pp. 87—96.

5 On a Tabulation of Reduced Binary Quadratic Forms of a Negative Determinant, by H. N. Wright. Pp. 97—114.

6 The Abelian Equations of the Tenth Degree irreducible in a given Domain of Rationality, by Kuschke. Pp. 115—162.

7 Abridged Tables of Hyperbolic Functions, by Pernot. Pp. 163—169.

8 A List of Oughtred's Mathematical Symbols with Historical Notes, by Cajori. Pp. 171—186.

9 On the History of Gunter's Scale and the Slide Rule during the Seventeenth Century, by Cajori. Pp. 187—209.

10 On a Birational Ttansformation Connected with a Pencil of Cubics, by A. R. Williams. Pp. 211—222.

11 Classification of Involutory Cubic Space Transformations, by F. R. Morris. Pp. 223—240.

12 A Set of Five Postulates for Boolean Algebras in Terms of the Operation "Exception," by J. S. Taylor. Pp. 241—248.

13 Flow of Electricity in a Magnetic Field, by Volterra, with the cooperation of Elena Freda. Pp. 249—320.

14 The Homogeneous Vector Function and Determinants of the P-th Class, by Barter. Pp. 321—343.

15 Involutary Quartic Transformations in Space of Four Dimensions, by Alderton. Pp. 345—358.

16 On the Indeterminate Cubic Equation, $X^3 + Dy^3 + D^2z^3 - 3\,Dxyz = 1$, by Wolfe. Pp. 359—369.

17 A Study and Classification of Ruled Quartic Surfaces by Means of a Point-to-Line-Transformation, by Wong. Pp. 371—387.

18 A Special Quartic Curve, by McFarland. Pp. 389—400.

19 A Study of Cubic Surfaces by Means of Involutory Cubic Space Transformations, by Pobanz. Pp. 401—423.

20 The Hyperspace Generalization of the Lines on the Cubic Surface, by Steed. Pp. 425—443.

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